Lecture 6: More on RCK

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Outline

- A brief review of Lecture 5
- Efficiency of the competitive equilibrium
- Competitive equilibrium of the RCK model
- Saddle path stability
- Final comments on neoclassical growth models

A Brief Review of Lecture 5 (1 of 5)

- Basic setup of the RCK model: Firms have access to a CRS production function

$$Y_t = F(K_t, A_t L_t^D) \tag{1}$$

- L identical individuals live forever, endowed with the initial capital stock, and endowed with one unit of labour per period, lifetime utility given by:

$$U(c_0, c_1, ...) = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2}$$

- Markets are perfectly competitive.

A Brief Review of Lecture 5 (2 of 5)

The competitive equilibrium:

- Firm's profit maximisation

$$r_t = f'(k_t) \tag{3}$$
$$w_t = f(k_t) - f'(k_t)k_t \tag{4}$$

- Individual's utility maximisation problem: An individual chooses sequences of savings to maximise her/his lifetime utility, subject to budget constraints in each period, taking as given prices and technology levels.

$$\max_{\{s_t\}_{t=0}^{\infty}} U(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \quad c_t + s_t = w_t A_t + (1 + r_t)s_{t-1}, \quad t = 0, 1, \ldots \tag{5}$$
$$s_{-1} = \frac{K_0}{L}$$

A Brief Review of Lecture 5 (3 of 5)

- The consumption Euler equation:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1} \tag{6}$$

- That is, the individual's utility maximisation requires that the MRS between consumption in any two adjacent periods be equal to the rate of return to savings for that period. The individual's problem is characterised by Eq.(5) and (6).
- Market clearing conditions:

$$L_t^D = L \tag{7}$$

$$K_{t+1} = Ls_t \tag{8}$$

A Brief Review of Lecture 5 (4 of 5)

- The transition equations: starts from the market clearing condition for capital market

$$s_t = \frac{K_{t+1}}{L} = A_{t+1}k_{t+1} \tag{9}$$

- Then substitute all other equations into the individual's budget constraint and the Euler equation:

$$\tilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1} \tag{10}$$

$$\frac{u'(A_t\tilde{c}_t)}{\beta u'(A_{t+1}\tilde{c}_{t+1})} = 1 + f'(k_{t+1}) \tag{11}$$

where $\tilde{c}_t = \frac{c_t}{A_t}$ and k_t denote consumption and capital per unit of effective labour respectively

- A steady state of k_t and $\tilde{c}$, denoted by k^* and $\tilde{c}^*$, satisfy:

$$\tilde{c}^* = f(k^*) + k^* - (1+g)k^* = f(k^*) - gk^*$$

$$\frac{u'(A_t\tilde{c}^*)}{u'(A_{t+1}\tilde{c}^*)} = \beta[1 + f'(k^*)]$$

A Brief Review of Lecture 5 (5 of 5)

- An example, suppose utility is CRRA, then Eq.(11) becomes

$$[\frac{\tilde{c}_{t+1}}{\tilde{c}_t}]^\theta = \frac{\beta[1 + f'(k_{t+1})]}{(1+g)^\theta} \tag{12}$$

- The transition equation, therefore, are given by Eq.(10) and (12). A steady state of k_t and $\tilde{c}$, denoted by k^* and $\tilde{c}^*$, satisfy:

$$\tilde{c}^* = f(k^*) - gk^* \tag{13}$$

$$\beta[1 + f'(k^*)] = (1+g)^\theta \tag{14}$$

  Notice that Eq.(13) and (14) determine a unique steady state value k^* and $\tilde{c}^*$.

- Once k_t and $\tilde{c}_t$ converge to their steady states, the economy reaches its stationary equilibrium, or balanced growth path, which exhibits similar properties as in the Solow-Swan model.

Evaluate Efficiency of the Competitive Equilibrium (1 of 5)

- The competitive equilibrium is an outcome determined by the interactions of firms and households in competitive markets. Again, we care about whether it is Pareto-efficient or Pareto-optimal.
- In general, Pareto-efficient allocations can be found by solving a social planner's problem: choosing feasible allocations to maximise some social welfare function.
- If the equilibrium allocation coincides with a solution to the social planner's problem, then the equilibrium is efficient.
- In the Diamond model, there are infinite number of generations: the initial old and all future generations. Hence, the social welfare function is not easy to define. So, we examined the efficiency of the competitive equilibrium by comparing k^* with k_{GR}.

Evaluate Efficiency of the Competitive Equilibrium (2 of 5)

- For the RCK model, since all individuals are identical, the social welfare function is well defined, which can be defined as a representative household's lifetime utility.
- Therefore, we will examine the efficiency of the competitive equilibrium by comparing it with the solution to the social planner's problem.
- **The social planner's problem**
  - The social planner's objective is to maximise a representative household's lifetime utility.
  - The only constraint that the social planner is subject to is the feasibility constraint. (i.e. must choose from allocations that satisfy the resource constraint of the economy)

Evaluate Efficiency of the Competitive Equilibrium (3 of 5)

- The aggregate resource constraint for the RCK economy:

$$F(K_t, A_t L) = C_t + (K_{t+1} - K_t) \tag{15}$$

- Dividing both sides by $A_t L$ to transform into units of per effective labour (i.e. intensive form):

$$f(k_t) = \tilde{c}_t + (1+g)k_{t+1} - k_t$$

or equivalently:

$$\tilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1} \tag{16}$$

- Therefore, the social planner's problem is

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} U(c_0, c_1, ...) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t. the resource constraint (16), $k_0 = \frac{K_0}{A_0 L}$ is given, $c_t = A_t \tilde{c}_t$

Evaluate Efficiency of the Competitive Equilibrium (4 of 5)

- FOC w.r.t. k_{t+1} is given by (similar as how we derived the FOCs of the household's utility maximisation problem):

$$\begin{aligned}\frac{\partial U}{\partial k_{t+1}} &= \beta^t u'(A_t\tilde{c}_t)A_t\frac{\partial \tilde{c}_t}{\partial k_{t+1}} + \beta^{t+1}u'(A_{t+1}\tilde{c}_{t+1})A_{t+1}\frac{\partial \tilde{c}_{t+1}}{\partial k_{t+1}} \\ &= \beta^t u'(A_t\tilde{c}_t)A_t[-(1+g)] + \beta^{t+1}u'(A_{t+1}\tilde{c}_{t+1})A_{t+1}[1+f'(k_{t+1})] \\ &= \beta^t\{-u'(A_t\tilde{c}_t)A_{t+1} + \beta u'(A_{t+1}\tilde{c}_{t+1})A_{t+1}[1+f'(k_{t+1})]\} = 0\end{aligned}$$

- We then have:

$$\frac{u'(A_t\tilde{c}_t)}{\beta u'(A_{t+1}\tilde{c}_{t+1})} = 1 + f'(k_{t+1}) \tag{17}$$

- The solution to the social planner's problem is characterised by Eq. (16) and (17). Notice that these two equations are exactly the transition equations (10) and (11) that characterise the competitive equilibrium.

Evaluate Efficiency of the Competitive Equilibrium (5 of 5)

- What does this imply?
- The competitive equilibrium coincides with the social planner's choice of allocation!
- The competitive equilibrium is Pareto-efficient.
- Reasons underlying this result is that conditions of the First Fundamental Welfare Theorem hold in the RCK model (i.e. competitive and complete markets, no externalities, finite number of agents).
- For RCK-type model, the competitive equilibrium can be found by solving the social planner's problem. The latter problem is much simpler.

The Golden Rule (1 of 4)

- How to find the golden rule k? Is k^* the same as k_{GR}?
- Recall that k_{GR} maximises steady state consumption.
- From Eq.(16), the stationary resource constraint is given by:

$$\tilde{c} = f(k) - gk \tag{18}$$

- Thus, k_{GR} solves $\max_k f(k) - gk$, (i.e. k_{GR} is determined by)

$$f'(k_{GR}) = g \tag{19}$$

- Is $k^* = k_{GR}$? Recall that with CRRA utility function and a general CRS production function, the unique k^* satisfies Eq.(14):

$$\beta[1 + f'(k^*)] = (1+g)^\theta \Rightarrow f'(k^*) = \frac{(1+g)^\theta}{\beta} - 1$$

The Golden Rule (2 of 4)

- To compare $\frac{(1+g)^\theta}{\beta} - 1$ and g, notice that an individual's lifetime utility can be written as:

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t u(c_t) &= \sum_{t=0}^{\infty} \beta^t \frac{(A_t \tilde{c}_t)^{1-\theta} - 1}{1-\theta} \\
&= \sum_{t=0}^{\infty} \beta^t \frac{[A_0 (1+g)^t \tilde{c}_t]^{1-\theta} - 1}{1-\theta} \\
&= \sum_{t=0}^{\infty} \frac{\beta^t (1+g)^{(1-\theta)t} (A_0 \tilde{c}_t)^{1-\theta}}{1-\theta} - \sum_{t=0}^{\infty} \frac{\beta^t}{1-\theta} \\
&= \sum_{t=0}^{\infty} \frac{[\beta (1+g)^{(1-\theta)}]^t (A_0 \tilde{c}_t)^{1-\theta}}{1-\theta} - \frac{1}{(1-\beta)(1-\theta)}
\end{aligned}$$

  The last equality follows from $\sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta}$

- Along a balanced growth path, $\tilde{c}_t = \tilde{c}^*$ for all t, thus if $\beta(1+g)^{(1-\theta)} \geq 1$, an individual can attain infinite lifetime utility, as a result his utility maximisation problem is not well-defined.

The Golden Rule (3 of 4)

- Therefore, we impose the restriction that

$$\beta(1+g)^{(1-\theta)} < 1$$

Equivalently $\frac{(1+g)^{\theta}}{\beta} - 1 > g$. Thus,

$$f'(k^*) = \frac{(1+g)^{\theta}}{\beta} - 1 > g = f'(k_{GR})$$

We, therefore, have

$$k^* < k_{GR} \tag{20}$$

- In the Solow-Swan and the Diamond model, k^* can be less than, equal or greater than k_{GR}. However, in the RCK model, the golden rule k_{GR} can not be attained as an equilibrium outcome.
- In other words, it can NOT be an equilibrium for the economy to converge to the balanced growth path that yields the maximum sustainable level of c.

The Golden Rule (4 of 4)

- Why?
- An increase in saving would decrease current consumption, but increase future capital stock and hence increase future consumption.
- But because individuals value present consumption more than future consumption (β is sufficiently less than 1, $\beta < \frac{1}{(1+g)^{1-\theta}}$), the benefit of increasing future consumption is eventually bounded.
- In other words, at some point when $k > k^*$, the trade-off between the intertemporal short-term sacrifice and the future gain is sufficiently unfavorable that further increasing saving reduces rather than raises lifetime utility.
- The optimal level of k for the economy, k^*, is called the modified golden rule capital stock.

Saddle Path Stability (1 of 4)

- We mentioned that the unique steady state (or stationary equilibrium, or balanced growth path) is saddle path stable. Now we give a bit explanations for this concept.
- Recall that in the Solow-Swan model, and the Diamond model with Cobb-Douglas production function and logarithmic utility, the unique balanced growth path is globally stable. That is, wherever the economy starts from, it follows the dynamics determined by the transition equation to converge to the balanced growth path.
- Recall that in the Solow-Swan model, the transition equation of capital per unit of effective labour has the form

$$k_{t+1} = g(k_t)$$

where $g(\cdot)$ is some concave function. With k_0 given, this transition equation determines the unique equilibrium path (trajectory) of k_t (first-order difference equation).

Saddle Path Stability (2 of 4)

- What is different in the RCK model? In the RCK model with CRRA utility, the transition equations are:

$$\tilde{c}_t = f(k_t) + k_t - (1+g)k_{t+1} \tag{21}$$

$$\left[\frac{\tilde{c}_{t+1}}{\tilde{c}_t}\right]^\theta = \frac{\beta[1+f'(k_{t+1})]}{(1+g)^\theta} \tag{22}$$

- We can rewrite the transition equations by substituting Eq.(21) into (22).

$$\left[\frac{f(k_{t+1}) + k_{t+1} - (1+g)k_{t+2}}{f(k_t) + k_t - (1+g)k_{t+1}}\right]^\theta = \frac{\beta[1+f'(k_{t+1})]}{(1+g)^\theta} \tag{23}$$

  Eq.(23) is second-order difference equation in k_t.

- With k_0 given, the transition equation (23) determines two trajectories (paths) of k_t, but only one trajectory converges to the unique steady state, while the other is not. We call the trajectory that converges to k^* a saddle path, and the steady state is saddle path stable.

Saddle Path Stability (3 of 4)

- The saddle path is the solution we are looking for. Then the question is how to rule out the other trajectory that also satisfies the transition equation? It can be ruled out by what we call the transversality condition (TVC), which is a necessary condition for infinite horizon optimisation problem.
- To understand the TVC, let us first consider the finite horizon case. If the model has finite horizon T (i.e. all individuals die at the end of period T), then a natural requirement for optimally is that $k_{T+1} = 0$.
  - Notice that $k_{T+1} < 0$ implies that individuals have unpaid debt when they die, or in other words individuals' lifetime consumption exceeds their lifetime income. This case is not allowed in the model – "No Ponzi game".
  - Also $k_{T+1} > 0$ is not optimal because the capital could be consumed to increase individuals' welfare.

- The TVC is the infinite horizon equivalent to this condition. In the current context, it is given by:

$$\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0 \tag{24}$$

where $\beta^T \frac{u'(c_T)}{u'(c_0)} k_{T+1} = \frac{1}{\frac{u'(c_0)}{\beta^T u'(c_T)}} k_{T+1} = \frac{k_{T+1}}{(1+r)^T}$. It states that the present discount value of capital stock goes to zero in the long run.

- To summarise, the unique balanced growth path in the RCK model is saddle path stable. The unique saddle path that converges to the balanced growth path is determined by the transition equations, together with two boundary conditions the initial capital stock and the TVC.

Final Comments on Neoclassical Growth Models (1 of 2)

- The Solow-Swan model provides a basic framework for one-sector neoclassical growth model. It treats the allocation of current income between consumption and investment in a mechanical way.
- The Diamond model and the RCK model can be seen as optimising versions of the Solow-Swan model. They aim to derive the consumption-investment decision from the decentralised behavior of intertemporal-utility-maximising households and perfectly competitive profit-maximising firms.
- The endogenous determined saving rate is a constant on the balanced growth path. Thus the Solow-Swan model's simplification of assuming an exogenous and fixed saving rate is not essential.

Final Comments on Neoclassical Growth Models (2 of 2)

- The behavior of the economy in both models is quite similar to the Solow-Swan model.
- The main predictions of the Solow-Swan model concerning sources of variations in per capita GDP over time and across countries still hold in both models.
- Therefore, in terms of growth theory, both models did not give much new insights other than those implied by the Solow-Swan model. The main contribution of the Diamond and the RCK models to the literature is that they have laid out the two basic frameworks for neoclassical growth theory.
- In particular, the representative agent framework in RCK model is extended to incorporate uncertainties (and other sectors) to address business cycle issue.